APPROXIMATION ALGORITHMS

THE PRIMAL-DUAL METHOD

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TODAY

- LINEAR PROGRAMMING DUALITY RECAP
- THE PRIMAL DUAL METHOD THROUGH CASE STUDIES:
    - SHORTEST PATHS
    - FEEDBACK VERTEX SETS
    - STEINER TREE

LINEAR PROGRAMMING DUALITY

INTUITION: SHOW BOUNDS ON OPT THROUGH AN OPTIMIZATION PROBLEM — ALSO AN LP

EXAMPLE

OBS:

$x_1, x_2 \in [0,1] \xrightarrow{\text{SINCE } x_1 \le 1 \text{ AND } x_2 \le 1} x_1 + x_2 \le 1 + 1 = 2$

$4x_1 + x_2 \le 2 \Rightarrow \frac{4}{7}x_1 + \frac{1}{7}x_2 \le \frac{2}{7}$

$x_1 + 2x_2 \le 1 \Rightarrow \frac{3}{7}x_1 + \frac{6}{7}x_2 \le \frac{3}{7}$

ADD: $x_1 + x_2 \le \frac{5}{7}$

OBJECTIVE:

MAXIMIZE $x_1 + x_2$

SET OF "TIGHT" CONSTRAINTS

$\downarrow$

BOUND ON MAXIMUM VALUE

DUALITY IN GENERAL

LP: MAXIMIZE $c^T \cdot x$
UNDER $Ax \leq b$
$x \geq 0$

$$\boxed{c, x \in \mathbb{R}^n, \; A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m}$$

DUAL: MINIMIZE $b^T y$
UNDER $A^T y \geq c$
$y \geq 0$

$$\boxed{b, y \in \mathbb{R}^m, \; A^T \in \mathbb{R}^{n \times m}, \; c \in \mathbb{R}^n}$$

WEAK DUALITY: FOR FEASIBLE x (WRT. LP) AND y (WRT. DUAL),

$$c^T x \leq b^T y.$$

IN PARTICULAR FEASIBLE AND BOUNDED

STRONG DUALITY: IF LP AND DUAL HAVE OPTIMAL SOLUTIONS x^* AND y^*, THEN $c^T x^* = b^T y^*$.

COMPLEMENTARY SLACKNESS CONDITIONS

IF X AND y ARE FEASIBLE SOLUTIONS TO AN LP AND ITS DUAL,
AND IF THE FOLLOWING CONDITIONS HOLD:

i) $x_j > 0 \Rightarrow (A^T y)_j = c_j$

ii) $y_i > 0 \Rightarrow (Ax)_i = b_i$

THEN X AND y ARE BOTH OPTIMAL.

TYPICAL GAME PLAN:

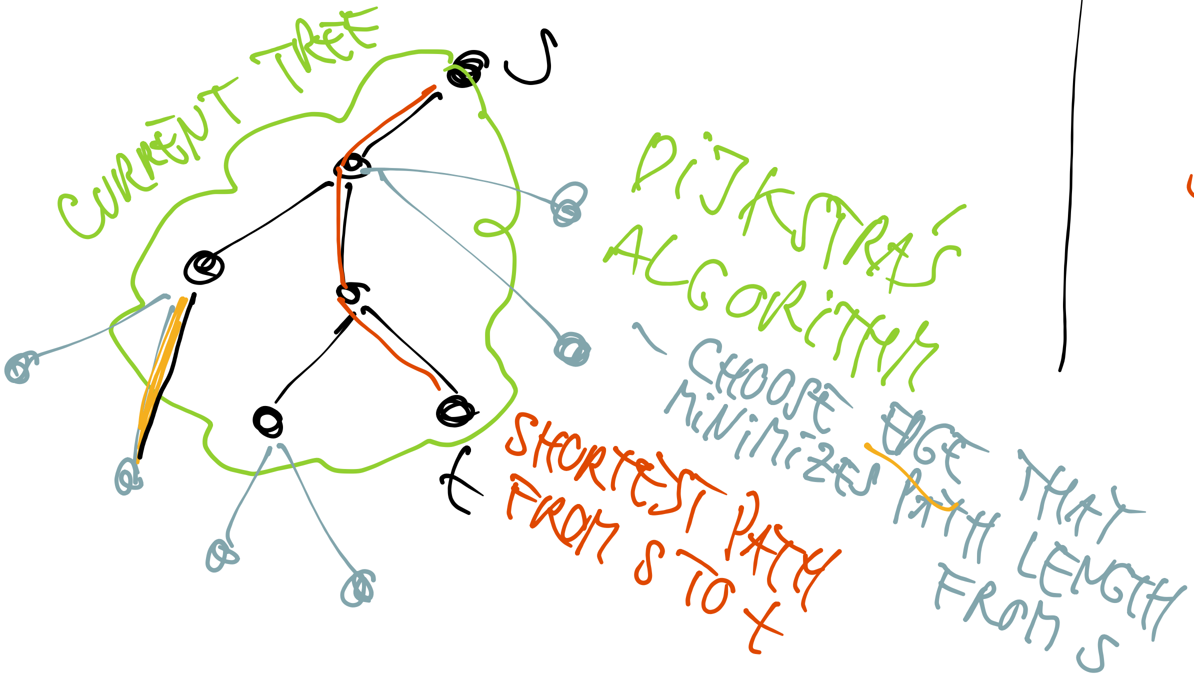
- TAKE PRIMAL INFEASIBLE X, DUAL FEASIBLE y
- IMPROVE DUAL SOLUTION S.T. SOME CONSTRAINT $(A^T y)_j \leq c_j$ HOLDS WITH = "TIGHT"
- MODIFY x_j
- REPEAT UNTIL X IS FEASIBLE

USE TIGHT CONSTRAINTS IN ANALYSIS

SHORTEST PATHS

INPUT: • GRAPH (V,E) WITH EDGE WEIGHT $c_e \geq 0$ FOR $e \in E$
• START VERTEX $s \in V$

OUTPUT: SHORTEST PATH TREE WITH ROOT s.



EQUIVALENT

INTEGER PROGRAM:

MINIMIZE $\sum_e c_e x_e$

SUBJECT TO $\sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V$

$x_e \in \{0,1\}$

WHERE $\delta(S)$ ARE EDGES CROSSING CUT $(S, V \setminus S)$.

SHORTEST PATH TREE:

$\{e \mid x_e = 1\}$

DUAL LP

MINIMIZE $\sum_e c_e x_e$

SUBJECT TO $\sum_{e \in \delta(S)} x_e \geq 1, \ \forall S \subseteq V$

$x_e \in \{0,1\}$

WHERE $\delta(S)$ ARE EDGES CROSSING CUT $(S, V \setminus S)$.

$\longrightarrow$

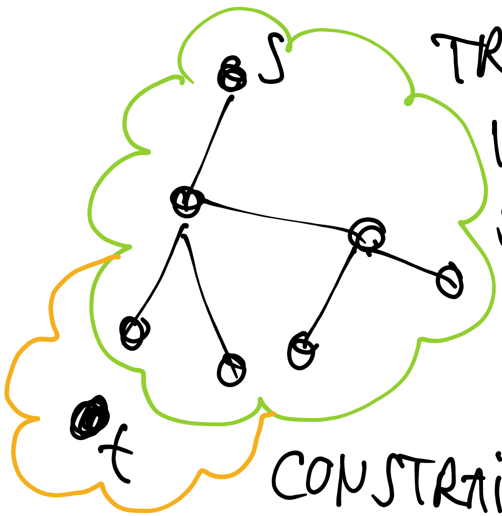
MAXIMIZE $\sum_{S \subseteq V, s \in S} y_S$

SUBJECT TO

$\sum_{S, |S \cap e| = 1} y_S \leq c_e, \ \forall e \in E$

$y_S \geq 0, \ \forall S \subseteq V, s \in S$

PAUSE & THINK: WHY IS THE EDGE ADDED THE SAME AS IN DIJKSTRA?



TREE F, VERTEX SET C

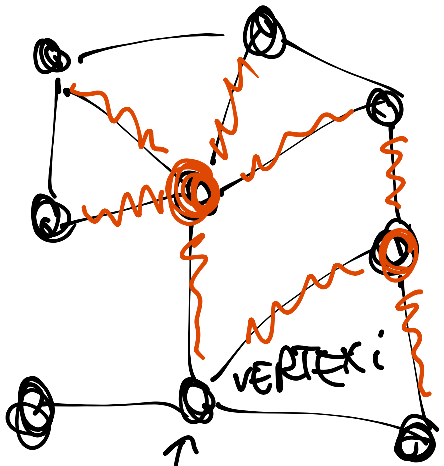
CONSTRAINT $\sum_{e \in \delta(C)} x_e \geq 1$ NOT SATISFIED

INVARIANT $\longrightarrow$ $y_C = 0$

CAN INCREASE y_C IN DUAL SOLUTION UNTIL SOME CONSTRAINT $\sum_{S, |S \cap e| = 1} y_S = c_e$ HOLDS WITH EQUALITY

ADD EDGE TO SHORTEST PATH TREE.

FEEDBACK VERTEX SET



WEIGHT $w_i \geq 0$

REMOVE SET $S \subseteq V$ OF MINIMUM WEIGHT TO MAKE GRAPH ACYCLIC

BOTH HAVE A HUGE NUMBER OF CONSTRAINTS BUT ONLY A TOOL FOR ANALYSIS

INTEGER PROGRAM:

MINIMIZE $\sum w_i x_i$

SUBJECT TO $\sum_{i \in C} x_i \geq 1$

FOR ALL CYCLES C IN (V, E)

$x_i \in \{0, 1\}$

DUAL:

MAXIMIZE $\sum_C y_C$

SUBJECT TO

$\sum_{C, i \in C} y_C \leq w_i \quad , \quad \forall i \in V$

$y_C \geq 0 \quad , \quad \forall \text{ CYCLES } C$

INTEGER PROGRAM:

MINIMIZE $\sum w_i x_i$

SUBJECT TO $\sum_{i \in C} x_i \geq 1$

FOR ALL CYCLES C IN (V,E)

$x_i \in \{0,1\}$

DUAL:

MAXIMIZE $\sum_C y_C$

SUBJECT TO

$\sum_{C,\, i \in C} y_C \leq w_i, \quad \forall i \in V$

$y_C \geq 0, \quad \forall$ CYCLES C

PRIMAL-DUAL ALGORITHM:

- START WITH $x=0$, $y=0$
  - $x=0$: PRIMAL INFEASIBLE
  - $y=0$: DUAL FEASIBLE

REPEAT UNTIL NO CYCLES LEFT:

- FIND CYCLE C WITH FEWEST VERTICES OF DEGREE ≥ 3
- INCREASE y_C UNTIL FOR SOME $i \in V$: $\sum_{C,\, i \in C} y_C = w_i$ ← "TIGHT DUAL CONSTRAINT"
- SET $x_i = 1$, REMOVE i FROM V

ANALYSIS:

$$\sum w_i x_i = \sum_{i,\, x_i=1} w_i = \sum_{i,\, x_i=1} \sum_{C,\, i \in C} y_C \overset{?}{\leq} f \sum_C y_C \leq f \cdot OPT.$$

NEXT PAGE

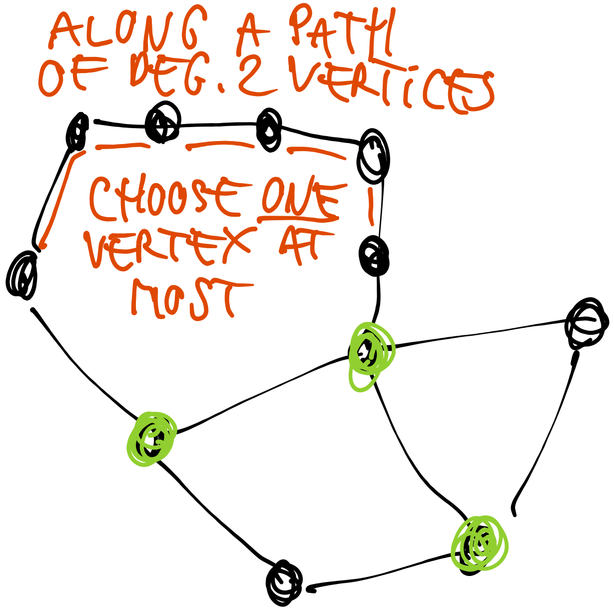
ANALYSIS, CONTINUED

$$\sum w_i x_i = \sum_{i,\, x_i=1} w_i = \sum_{i,\, x_i=1} \sum_{C,\, i\in C} y_C = \sum_{\text{CYCLES } C} |S\cap C|\, y_C, \quad \text{WHERE } S=\{i \mid x_i=1\}$$

$$\le O(\log n) \sum_{\text{CYCLES } C} y_C$$

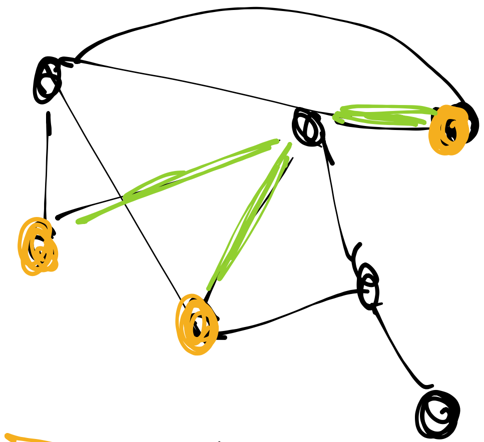
$$\le O(\log n)\cdot OPT.$$

NOT BEST POSSIBLE, SEE WS SECTION 14.2 FOR A 2-APPROX.



AMONG HIGH-DEGREE (≥3) VERTICES, CAN ALWAYS FIND O(log n) LENGTH CYCLE (NOT COUNTING DEGREE 2 VERTICES)

STEINER TREE PROBLEM



UNDIRECTED WEIGHTED GRAPH (V, E)

$T \subseteq V$

TERMINAL VERTICES

GOAL: TREE F CONNECTING NODES IN T, MINIMIZING WEIGHT

INTEGER PROG. FORMULATION:

WEIGHT OF EDGE e

MINIMIZE $\sum_{e \in E} c_e x_e$

SUBJECT TO $x_e \in \{0,1\}, \forall e \in E$

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \in P = \{S \mid |S \cap T| \notin \{0, |T|\}\}$$

CONTAINS SOME BUT NOT ALL TERMINALS

PAUSE TO THINK:

WHY IS THIS PROBLEM EQUIVALENT TO THE STEINER TREE PROBLEM?

PRIMAL LP RELAXATION

MINIMIZE $\sum_{e \in E} c_e x_e$ ← WEIGHT OF EDGE e

SUBJECT TO $\cancel{x_e \in \{0,1\}}$ $x_e \geq 0$, $\forall e \in E$

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \in P = \{S \mid |S \cap T| \notin \{0, |T|\}\}$$

DUAL LP

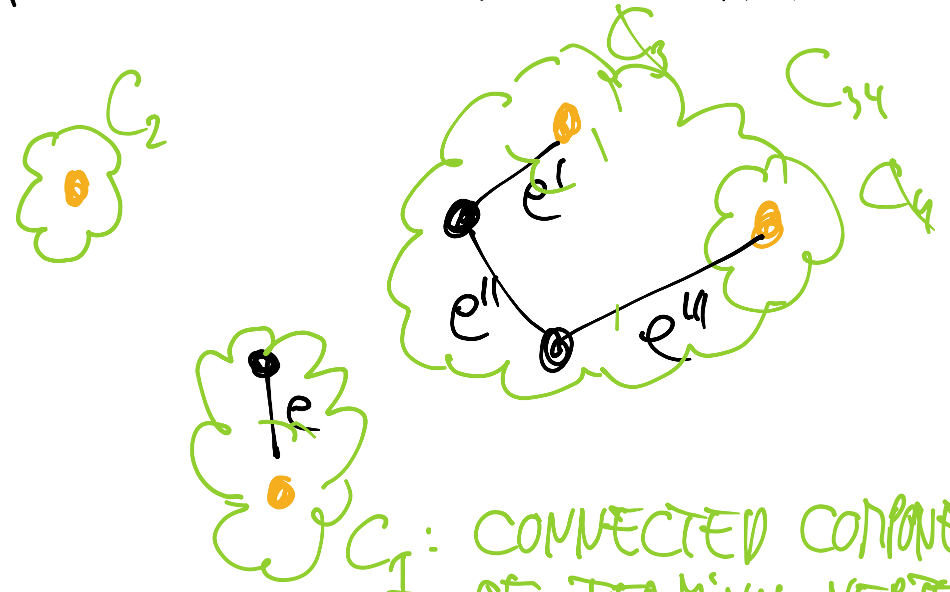
MAXIMIZE $\sum_{S \in P} y_S$

SUBJECT TO

$$\sum_{S,\, e \in \delta(S)} y_S \leq c_e, \; \forall e \in E$$

$$y_S \geq 0, \; \forall y \in P$$

PRIMAL-DUAL ALGORITHM:



C_1: CONNECTED COMPONENT OF TERMINAL VERTEX 1

REPEAT UNTIL TERMINALS ARE CONNECTED

- INCREASE ALL OF $y_{C_1}, y_{C_2}, y_{C_3}, \ldots$ UNTIL SOME CONSTRAINT SATISFIES $\sum_{S,\, e \in \delta(S)} y_S = c_e$ ← WILL NEVER CREATE A CYCLE, ADD TO F
- ADD e TO ONE OF THE COMPONENTS (IT MIGHT CONNECT TWO OF THEM)
- REMOVE AND "DANGLING" EDGES

ANALYSIS (SKETCH)

BUT: CAN SHOW INEQUALITY BY INDUCTION ON $|F|$

$$\sum_{e\in F} c_e = \sum_{e\in F}\sum_{S,\, e\in\delta(S)} y_S = \sum_{S\in\mathcal{P}} |F\cap\delta(S)|\, y_S \leq 2\sum_{S\in\mathcal{P}} y_S \leq 2\cdot OPT \qquad \blacksquare$$

NOT TRUE THAT $|F\cap\delta(S)|\leq 2$

BYRKA ET AL. (2010): CAN GET APPROX. FACTOR $\approx \ln(4) = 1.386\ldots$

IN BOOK:

MORE GENERAL STEINER FOREST PROBLEM WHERE PAIRS $(s_1,t_1), (s_2,t_2)\ldots$ MUST BE CONNECTED